A square matrix has an associated number that represents summary information about that matrix – a scalar we call the determinant. As we have seen, the determinant of **A** is denoted by |**A**|.

The determinant provides useful information about the permissible operations on a matrix. If the determinant is equal to zero, then the matrix is *singular* and we cannot compute its inverse.

In multivariate statistics, we are most interested in computing the determinant of sigma, **Σ**, the variance-covariance matrix. This matrix is square and symmetric, which means that the matrix **Σ** and its transpose **Σ'** are identical. We have seen this matrix many times. Recall that correlations are standardized covariances, so the sample variance-covariance matrix **Σ** can be represented as:

$$\boldsymbol{\Sigma} = \begin{bmatrix} s_1^2 & s_1 s_2 r_{12} & \cdots & s_1 s_p r_{1p} \\ s_2 s_1 r_{21} & s_2^2 & \cdots & s_2 s_p r_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ s_p s_2 r_{p2} & s_p s_2 r_{p2} & \cdots & s_p^2 \end{bmatrix}$$ where *s* is the standard deviation and *r* is the correlation.

As we noted earlier, the determinant of **Σ** is referred to as the ***generalized variance***.

Characteristic Equation

Every square matrix has an associated *characteristic equation*, which is formed by subtracting a specific value, lambda: λ, from each diagonal element of the matrix, so that the determinant of the resulting matrix is equal to zero.

Consider a simple 2×2 matrix **A**. We would attempt to identify this specific value so that the following is true:

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

For a matrix of order *p*, there are potentially *p* different values for λ that will satisfy this equality. The values of λ that solve this equality are called the *eigenvalues* of the matrix.

Associated with each eigenvalue is a vector, here we will use $\underline{v}$, called the eigenvector. Eigenvectors satisfy the equation $\mathbf{A}\underline{v} = \lambda\underline{v}$.

If all eigenvalues are placed in the principal diagonal of a diagonal matrix **L**, then the relation, **AV** = **VL**, also holds for matrices.

This equation provides the eigenstructure of **A**. We rely heavily on the eigenstructures in multivariate procedures.

[1] http://www.itl.nist.gov/div898/handbook/pmc/section5/pmc532.htm

Example

We can consider an example using a correlation matrix, for simplification, but also for numerical efficiency. Recognize that correlations are standardized covariances, placing them all on the same scale (actually, they are considered to be scale free).

In this example, I will use SPSS matrix notation to complete all operations. Here we can ask SPSS to compute the eigenvalues and eigenvectors, then we can use them to verify the relations described above.

Consider the correlation matrix **R**:

$$\mathbf{R} = \begin{bmatrix} 1.00 & .50 & .60 \\ .50 & 1.00 & .70 \\ .60 & .70 & 1.00 \end{bmatrix}$$

```
compute R = {1, .5, .6; .5, 1, .7; .6, .7, 1}.
print R.

R
   1.000000000    .500000000    .600000000
    .500000000   1.000000000    .700000000
    .600000000    .700000000   1.000000000
```

Estimate the determinant of **R**:

$$|\mathbf{R}| = \begin{vmatrix} 1.00 & .50 & .60 \\ .50 & 1.00 & .70 \\ .60 & .70 & 1.00 \end{vmatrix}$$

```
compute dR = det(R).
print dR.

DR
   .3200000000
```

In SPSS, we can use a shorthand tool to estimate
(a) the eigenvectors of **R** (<u>v</u> or **V**, normalized unit-length vectors) and
(b) the eigenvalues (lambdas).

In this case from our notation above for the eigenstructure, this matrix of eigenvectors is

$$\mathbf{V} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

and <u>Lambda</u> contains the eigenvalues:

$$\underline{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

```
call eigen(R,v,lambda).
print v.
print lambda.

V
  -.5434581644   .8059688345   .2346645720
  -.5791257728  -.5623509647   .5902327777
  -.6076730723  -.1848665203  -.7723715472

LAMBDA
   2.203711293
    .513510476
    .282778231
```

We can demonstrate the orthonormal property of the vectors by computing **V'V**. From the *Definitions* handout, we defined **V** as an orthogonal matrix if **V'V** = **I**. Here we see **V'V** has rank = 3, and being of full rank, has linearly independent rows and columns.

```
compute tvv = t(v)*v.
print tvv.

TVV
   1.000000000    .000000000    .000000000
    .000000000   1.000000000    .000000000
    .000000000    .000000000   1.000000000
```

To continue, we can estimate a scaling for **V**. We do this by making a diagonal matrix with the square roots of <u>Lambda</u>, the eigenvalues.

```
Compute s = mdiag(sqrt(lambda)).
Print s.

S
   1.484490247    .000000000    .000000000
    .000000000    .716596453    .000000000
    .000000000    .000000000    .531768964
```

Now we can rescale the vectors in **V**:

cL = VS =

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix}$$

$$= \begin{bmatrix} e_{11}\sqrt{\lambda_1} & e_{12}\sqrt{\lambda_2} & e_{13}\sqrt{\lambda_3} \\ e_{21}\sqrt{\lambda_1} & e_{22}\sqrt{\lambda_2} & e_{23}\sqrt{\lambda_3} \\ e_{31}\sqrt{\lambda_1} & e_{32}\sqrt{\lambda_2} & e_{33}\sqrt{\lambda_3} \end{bmatrix}$$

```
compute cL = v*s.
Print cL.

CL
  -.8067583446   .5775544079   .1247873363
  -.8597065614  -.4029787065   .3138674725
  -.9020847491  -.1324746927  -.4107232173
```

Here we create the diagonal matrix of eigenvalues. This is **L** from our notation above for the eigenstructure. Recall that cL = v*s, so **L** = (**cL**)'(**cL**) = (**VS**)'(**VS**) = **S'V'VS**. Since **V'V** = **I** because **V** is orthonormal, then **S'V'VS** = **S'IS** = **S'S**.
Since **S** is a diagonal matrix of elements $\sqrt{\lambda}$, **S'S** = a diagonal matrix of elements λ.

```
Compute L = t(cL)*cL.
Print L.

L
   2.203711293    .000000000    .000000000
    .000000000    .513510476    .000000000
    .000000000    .000000000    .282778231
```

One thing to notice is the trace of this diagonal matrix: tr(**D**) = $\sum a_{ii}$.
This is *p*, the number of possible eigenvalues and eigenvectors. It is also the rank of **A**.

```
compute trL = trace(L).
print trL.

TRD
   3.000000000
```

<table>
<tr>
<td>We can reproduce the correlation matrix from the CL matrix of rescaled V (the eigenvectors).

R = (CL)(CL)'
= (VS)(VS)' = VSS'V' = VLV'

This is a matrix of eigenvalues pre and post-multiplied by corresponding elements of the eigenvectors. With this information, we can reproduce the correlation matrix, since the correlations were initially decomposed into eigenvalues and eigenvectors.

Notice that R = VLV'

This is the complete spectral decomposition of R.</td>
<td><pre>compute r2 = cL*t(cL).
print r2.

R2
   1.000000000   .500000000   .600000000
    .500000000   1.000000000   .700000000
    .600000000   .700000000   1.000000000</pre></td>
</tr>
<tr>
<td>Using the correlation matrix, and the eigenvector matrix V, we can estimate the left-hand side of the eigenstructure equation RV or AV.</td>
<td><pre>compute es1 = R*v.
print es1.

ES1
  -1.197624894   .413873440   .066358032
  -1.276226006   -.288773112   .166904981
  -1.339136012   -.094930895   -.218409860</pre></td>
</tr>
<tr>
<td>Using the eigenvector matrix V and the eigenvalue diagonal matrix L, we can estimate the right-hand side of the eigenstructure equation VL.</td>
<td><pre>compute es2 = v*L.
print es2.

ES2
  -1.197624894   .413873440   .066358032
  -1.276226006   -.288773112   .166904981
  -1.339136012   -.094930895   -.218409860</pre></td>
</tr>
<tr>
<td>We notice that AV = VL, the complete eigenstructure of A, or R in our case.</td>
<td></td>
</tr>
</table>